

GOOD STEIN NEIGHBORHOOD BASES AND REGULARITY OF THE $\bar{\partial}$ -NEUMANN PROBLEM ¹

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1. INTRODUCTION

The purpose of this note is to initiate a study of global regularity of the $\bar{\partial}$ -Neumann problem on a smooth bounded pseudoconvex domain Ω in terms of properties of Stein neighborhood bases of the closure $\bar{\Omega}$ of the domain. If $\bar{\Omega}$ is merely assumed to admit a Stein neighborhood basis, global regularity does not follow. Indeed, the closures of the worm domains ([9]) that wind only a little do admit Stein neighborhood bases (see e.g. [1], Theorem 5.1), yet global regularity fails ([6, 7]). On the other hand, if $\bar{\Omega}$ admits the ‘best possible’ Stein neighborhood basis, then the $\bar{\partial}$ -Neumann problem is globally regular on Ω . Namely, assume that the sublevel sets of the Euclidean distance to the boundary (outside $\bar{\Omega}$), $\Omega_\varepsilon := \{z \in \mathbb{C}^n / \text{dist}(z, \bar{\Omega}) < \varepsilon\}$, are pseudoconvex for small enough $\varepsilon > 0$. Fix such an ε . Then $b\Omega$ is a level set of the distance to the boundary of Ω_ε , hence a level set of minus the logarithm of this distance. Because Ω_ε is pseudoconvex, the latter function is plurisubharmonic in Ω_ε . Ω thus admits a plurisubharmonic defining function, and consequently the $\bar{\partial}$ -Neumann problem is globally regular on Ω ([2]). We show in section 2 (Theorem 1) that, more generally, if $\bar{\Omega}$ admits a ‘sufficiently nice’ Stein neighborhood basis, then global regularity holds. Part of the argument again exploits the plurisubharmonicity of minus the logarithm of the boundary distance on a pseudoconvex domain. In section 3 we briefly discuss (as it turns out to be) a generalization: global regularity holds as soon as the weakly pseudoconvex directions at boundary points are limits, from inside, of weakly pseudoconvex directions of level sets of the boundary distance (Theorem 2).

On a bounded pseudoconvex domain, the $\bar{\partial}$ -Neumann operator N_q , ($1 \leq q \leq n$) is the inverse of the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ acting on $(0, q)$ -forms. For background on the $\bar{\partial}$ -Neumann problem, we refer the reader to [10] and to the recent survey [4].

2. DOMAINS WITH GOOD STEIN NEIGHBORHOOD BASES

Let Ω be a bounded smooth (C^∞) pseudoconvex domain in \mathbb{C}^n . For $\varepsilon > 0$, denote by Ω_ε the outside sublevel sets of the Euclidean distance to $\bar{\Omega}$, i.e. $\Omega_\varepsilon := \{z \in \mathbb{C}^n / \text{dist}(z, \bar{\Omega}) < \varepsilon\}$. For $\varepsilon > 0$ small enough, Ω_ε is also smooth. Moreover, if $n(z)$ denotes the (real) outward unit normal to $b\Omega$,

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then $b\Omega_\varepsilon = \{z + \varepsilon n(z)/z \in b\Omega\}$. $W_{(0,q)}^s(\Omega)$ is the space of $(0,q)$ -forms with coefficients in $W^s(\Omega)$, with the usual inner product, where $W^s(\Omega)$ is the \mathcal{L}^2 -Sobolev space of order s . We use the ‘little-o’ notation in the usual sense.

Theorem 1. *Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^n . Assume there is a function $r(\varepsilon)$ with $1 - r(\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \rightarrow 0^+$ such that for ε small enough, there exists a pseudoconvex domain $\tilde{\Omega}_\varepsilon$ with $\Omega_{r(\varepsilon)\varepsilon} \subseteq \tilde{\Omega}_\varepsilon \subseteq \Omega_\varepsilon$. Then the $\bar{\partial}$ -Neumann operators N_q are continuous on $W_{(0,q)}^s(\Omega)$, for $s \geq 0$, $1 \leq q \leq n$.*

Remark 1. The case mentioned in the introduction above, when the Ω_ε themselves are pseudoconvex for small ε , corresponds to the case $1 - r(\varepsilon) \equiv 0$ in Theorem 1. This situation was studied in [14]. The necessary and sufficient condition is as follows. Denote by $H_\rho(z)$ the real Hessian of a defining function ρ , and by J (the matrix of) the orthogonal transformation on $\mathbb{R}^{2n} \approx \mathbb{C}^n$ corresponding to multiplication by $\sqrt{-1}$ (on \mathbb{C}^n). Assume now ρ is normalized so that $|\nabla \rho| = 1$ on $b\Omega$. Then the domains Ω_ε are pseudoconvex for $\varepsilon > 0$ small enough if and only if the quadratic form given by the matrix $H_\rho(z)(1 + \varepsilon H_\rho(z))^{-1} + J^T H_\rho(z)(1 + \varepsilon H_\rho(z))^{-1} J$ is positive semi-definite on the complex tangent space to $b\Omega$ at z for $z \in b\Omega$ and ε small enough. Here J^T denotes the transpose of J . This is proved in Proposition 1 in [14]. Note that the leading term (the 0-th order term) in the expansion with respect to ε is $H_\rho(z) + J^T H_\rho(z) J$, which agrees with the Levi-form at z (see e.g. [5], pp. 166–168 for a detailed comparison of the Levi-form of a hypersurface with its second fundamental form).

Remark 2. The closure of Ω is called uniformly H -convex if there exists a constant c , $0 < c < 1$, such that for all small enough ε , there is a pseudoconvex domain $\tilde{\Omega}_\varepsilon$ with $\Omega_{c\varepsilon} \subseteq \tilde{\Omega}_\varepsilon \subseteq \Omega_\varepsilon$. This condition is useful in proving results concerning approximation by functions holomorphic in a neighborhood of $\bar{\Omega}$, see e.g. [8]. The condition used in Theorem 1 is a restricted version of uniform H -convexity. Note that uniform H -convexity is not strong enough to give global regularity: the worm domains that wind only a little (that is, the parameter r in [9] is sufficiently close to 1) are uniformly H -convex. More specifically, for $\eta > 1$ fixed, there exists $r(\eta)$ such that if the parameter r in [9] satisfies $1 \leq r < r(\eta)$, then there is a defining function ρ for Ω_r (Ω_r is as in [9]) such that $(\rho_+)^n$ is plurisubharmonic in a neighborhood of $\bar{\Omega}_r$. Here, $\rho_+(z) := \max\{\rho(z), 0\}$. This follows from the proof of Theorem 3.2 in [13] (see also Remark 2 following that proof) together with the fact that the boundary of Ω_1 is B -regular. (The weakly pseudoconvex boundary points form a circle, so that the boundary is a countable union of B -regular sets, hence is itself B -regular. See [12] for

details on B -regularity.)

The proof of Theorem 1 results from the existence of vector fields that almost commute with $\bar{\partial}$: for every positive ε there exists a vector field X_ε of type $(1,0)$ with coefficients in $C^\infty(\bar{\Omega})$ such that $X\rho = 1$ on $b\Omega$ and such that when $1 \leq j \leq n$, the Hermitian inner product of the commutator $\left[X_\varepsilon, \frac{\partial}{\partial \bar{z}_j}\right]$ with the complex normal has modulus less than ε on the boundary. Here, ρ is a defining function for Ω . It is shown in [2, 3] that the existence of such a family of vector fields implies global regularity (i.e. the conclusion of Theorem 1). Moreover, the proof of the lemma in [2] shows that the crucial point is to have vectors fields with the above commutator property, but with the commutators taken with tangential fields \bar{Z} of type $(0,1)$ and evaluated only at points $z \in b\Omega$ where $\bar{Z}(z)$ is in the null space of the Levi form at z . Once a family of vector fields with this restricted commutator property is in hand, it can be modified by suitable complex tangential fields to obtain a family that satisfies the full commutator conditions (see [2], pp. 85–86 for details).

Let $\rho(z)$ be a defining function for Ω that near the boundary equals plus or minus the boundary distance, depending on whether z is outside Ω or inside. Note that if $z \in b\Omega$, and $w \in \mathbb{C}^n$ is contained in the complex tangent space to $b\Omega$ at z , then for $|\varepsilon|$ small, w is also contained in the complex tangent space to $b\Omega_\varepsilon$ at $z + \varepsilon n(z)$, where $n(z)$ denotes the outward unit normal to $b\Omega$, and $\Omega_\varepsilon := \{z \in \mathbb{C}^n / \rho < \varepsilon\}$. (The real tangent spaces at z and $z + \varepsilon n(z)$ agree, hence so do the complex tangent spaces.) Denote by $\mathcal{L}_\rho(z)(w, \bar{w})$ the Levi form of ρ (i.e. the complex Hessian) at z applied to $w \in \mathbb{C}^n$. Let $n(z) := 2 \left(\frac{\partial \rho}{\partial \bar{z}_1}, \dots, \frac{\partial \rho}{\partial \bar{z}_n} \right)$, i.e. $n(z)$ is the unit normal; we also denote the associated $(1,0)$ vector field, $2 \sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial z_j}$, by $n(z)$. If $Z(z)$ is a tangential field of type $(1,0)$, computing the normal $(1,0)$ component of $[n, \bar{Z}]$ gives, up to the factor 4,

$$\begin{aligned} \sum_{j=1}^n \bar{Z} \left(\frac{\partial \rho}{\partial \bar{z}_j} \right) \frac{\partial \rho}{\partial z_j} &= - \sum_{j=1}^n \frac{\partial \rho}{\partial \bar{z}_j} \bar{Z} \left(\frac{\partial \rho}{\partial z_j} \right) \\ &= - \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \frac{\partial \rho}{\partial \bar{z}_j} \bar{\zeta}_k = -\frac{1}{2} \mathcal{L}_\rho(n, \bar{Z}), \end{aligned} \tag{1}$$

where $Z = (\zeta_1, \dots, \zeta_n)$. We have used that $\sum_{j=1}^n \frac{\partial \rho}{\partial z_j} \frac{\partial \rho}{\partial \bar{z}_j} \equiv 1/4$ on $b\Omega$. Consequently, by our discussion above, it suffices to show that under the assumptions of Theorem 1

$$\mathcal{L}_\rho(z)(n, \bar{w}) = 0, \quad z \in b\Omega, w \in N_z, \tag{2}$$

where N_z denotes the null space of the Levi form at z . ($2n(z)$ will satisfy the normalization $2n(z)(\rho) = 1$.)

Consider now a weakly pseudoconvex boundary point z_0 and $0 \neq w \in N_{z_0}$. We claim that

$$\frac{d}{d\varepsilon} \mathcal{L}_\rho(z_0 + \varepsilon n(z_0))(w, \overline{w})|_{\varepsilon=0} = 0. \quad (3)$$

First note that since the interior sublevel sets of ρ are pseudoconvex (by the discussion in the introduction), and $\mathcal{L}_\rho(z_0)(w, \overline{w}) = 0$, this derivative is less than or equal to zero. Call its value a ; so $a \leq 0$. The assumption that $a < 0$ leads to a contradiction as follows. There is $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon < \varepsilon_0$, $\mathcal{L}_\rho(z_0 + \varepsilon n(z_0))(w, \overline{w}) \leq \frac{a}{2}\varepsilon$. Thus the Levi form of ρ at $z_0 + \varepsilon n(z_0)$ has a negative eigenvalue, and this implies that analytic functions on Ω_ε extend past the boundary point $z_0 + \varepsilon n(z_0)$, see e.g. [11], Theorem 2.6.13, where a version for extension of CR-functions is proved. Inspection of the proof shows that there is a constant $c > 0$, independent of ε (for $0 < \varepsilon \leq \varepsilon_0$), such that one has extension in the direction of the normal of at least $c(|a|\varepsilon)^3$. Applying this to the situation in Theorem 1, at the point $z_0 + r(\varepsilon)\varepsilon n(z_0)$ (i.e. replacing ε by $r(\varepsilon)\varepsilon$), and taking into account the pseudoconvexity of the domain $\widetilde{\Omega}_\varepsilon$, gives

$$r(\varepsilon)\varepsilon + c(|a|r(\varepsilon)\varepsilon)^3 \leq \varepsilon \quad (4)$$

(since the extension cannot go past $b\widetilde{\Omega}_\varepsilon$, hence not past $b\Omega_\varepsilon$). (4) implies

$$c|a|^3(r(\varepsilon))^3\varepsilon^2 \leq 1 - r(\varepsilon), \quad (5)$$

which contradicts the assumption in Theorem 1 that $1 - r(\varepsilon) = o(\varepsilon^2)$ (if $a < 0$). This establishes (3).

We next exploit the plurisubharmonicity of $-\log(-\rho)$ on Ω , near the boundary (where ρ agrees with minus the boundary distance). The nonnegativity of the complex Hessian of $-\log(-\rho)$ gives, after multiplication by $-\rho$:

$$\sum_{j,k} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) - \frac{1}{\rho(z)} \frac{\partial \rho}{\partial z_j}(z) \frac{\partial \rho}{\partial \bar{z}_k}(z) \right) w_j \overline{w}_k \geq 0, \quad w \in \mathbb{C}^n \quad (6)$$

for $z \in \Omega$ close enough to $b\Omega$. This positivity allows us to use Cauchy-Schwarz to estimate a mixed term (as in (2)!) by the quadratic terms. Moreover, the mixed term in the form in (6) is the same as the one in \mathcal{L}_ρ . To make this precise, denote the Hermitian form on the left-hand side of (6) by $Q_\rho(z)(w, \overline{w})$. Fix a weakly pseudoconvex boundary point z_0 , and let $w = (w_1, \dots, w_n) \in N_{z_0}$.

Then for $\varepsilon > 0$ small, we have

$$\begin{aligned}
|\mathcal{L}_\rho(z_0 - \varepsilon n(z_0))(n(z_0), w)| &= 2 \left| \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z_0 - \varepsilon n(z_0)) \left(\frac{\partial \rho}{\partial \bar{z}_j}(z_0) \right) \bar{w}_k \right| \\
&= 2 |Q_\rho(z_0 - \varepsilon n(z_0))(n(z_0), \bar{w})| \\
&\leq 2 Q_\rho(z_0 - \varepsilon n(z_0))(n(z_0), \overline{n(z_0)})^{1/2} Q_\rho(z_0 - \varepsilon n(z_0))(w, \bar{w})^{1/2} \\
&= 2 Q_\rho(z_0 - \varepsilon n(z_0))(n(z_0), \overline{n(z_0)})^{1/2} \mathcal{L}_\rho(z_0 - \varepsilon n(z_0))(w, \bar{w})^{1/2}.
\end{aligned} \tag{7}$$

Note that the normal to $b\Omega_{-\varepsilon}$ at $z_0 - \varepsilon n(z_0)$ equals $n(z_0)$, and that w also belongs to the complex tangent space to $b\Omega_{-\varepsilon}$ at $z_0 - \varepsilon n(z_0)$, so that $\sum_{k=1}^n \frac{\partial \rho}{\partial \bar{z}_k}(z_0 - \varepsilon n(z_0)) \bar{w}_k = 0$. Consequently, $\mathcal{L}_\rho(z_0 - \varepsilon n(z_0))(\cdot, \bar{w}) = Q_\rho(z_0 - \varepsilon n(z_0))(\cdot, \bar{w})$. The quadratic terms in the last term in (7) are now easily estimated. First, $Q_\rho(z_0 - \varepsilon n(z_0))(n(z_0), \overline{n(z_0)})$ is dominated by $1/\varepsilon$, just by the definition of Q_ρ . Second, $\mathcal{L}_\rho(z_0 - \varepsilon n(z_0))(w, \bar{w})$ is dominated by ε^2 , in view of (3) and the fact that $w \in N_{z_0}$ (i.e. $\mathcal{L}_\rho(z_0)(w, \bar{w}) = 0$). Inserting these two estimates into (7) and letting $\varepsilon \rightarrow 0^+$ yields (2). This completes the proof of Theorem 1.

Remark 3. The argument in the last part of the proof of Theorem 1 that exploits the plurisubharmonicity of $-\log(-\rho)$ is of interest in part because of its potentially broader scope, but in the situation of Theorem 1, one can exploit the work in [14] to obtain an alternative proof (as well as further information). It is shown in [14] that $\mathcal{L}_\rho(z_0 + \varepsilon n(z_0))(w, \bar{w})$ can be conveniently expressed in terms of $H_\rho(z_0)$, the *real* Hessian of ρ at z_0 . In fact, the expression is the one quoted in Remark 1 above. (One can always express the Levi form in terms of the second fundamental form, see [5], pp. 166–168. The point of [14] is that in the case of the level surfaces of the boundary distance, the (matrix corresponding to the) second fundamental form of the level surface is easily expressed in terms of the (matrix corresponding to the) second fundamental form of the boundary itself). From this expression, the derivative in (3) is computed to be

$$\frac{d}{d\varepsilon} \mathcal{L}_\rho(z_0 + \varepsilon n(z_0))(w, \bar{w})|_{\varepsilon=0} = -(\|H_\rho(z_0)\omega\|^2 + \|H_\rho(z_0)J\omega\|^2), \tag{8}$$

where ω denotes the vector in \mathbb{R}^{2n} corresponding to $w \in \mathbb{C}^n$ under the usual identification. Thus if this derivative vanishes, we obtain

$$H_\rho(z_0)\omega = H_\rho(z_0)J\omega = 0, \quad \omega \in N_{z_0}. \tag{9}$$

That is, (real) derivatives of the normal in directions in the null space of the Levi form vanish. By the discussion above (see in particular (1)), this suffices to prove the theorem. This approach gives

some additional information. In particular, in the situation of Theorem 1, if $z_0 \in b\Omega$ and $w \in N_{z_0}$, then $\mathcal{L}_\rho(z_0 + \varepsilon n(z_0))(w, \bar{w}) \equiv 0$ for ε close enough to 0. It is instructive to compare this with the necessary and sufficient condition for pseudoconvexity of Ω_ε discussed in Remark 1 above; compare also the discussion on p. 401 in [14], in particular Corollary 2.

3. A GENERALIZATION

The arguments in the proof of Theorem 1 show that global regularity holds if (3) holds, but (see Remark 3 above), if (3) holds, then actually $\mathcal{L}_\rho(z_0 + \varepsilon n(z_0))(w, \bar{w}) \equiv 0$ for $|\varepsilon|$ small and $w \in N_{z_0}$. In particular, the weakly pseudoconvex directions at boundary points are limits, along the normal, of weakly pseudoconvex directions at points of interior level sets of the boundary distance. The following is thus a generalization (also of Theorem 1). We keep the notation from above, so that ρ is still the defining function given by $+$ or $-$ the boundary distance (near $b\Omega$).

Theorem 2. *Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^n with the property that for each pair $(z, w) \in b\Omega \times \mathbb{C}^n$ with $w \in N_z$, there is a sequence $\{(z_n, w_n)\}_{n=1}^\infty$ in $\Omega \times \mathbb{C}^n$ such that $(z_n, w_n) \rightarrow (z, w)$ as $n \rightarrow \infty$, and for all n , w_n is in the null space of the Levi-form at z_n of the level set of ρ through z_n . Then the $\bar{\partial}$ -Neumann operators N_q , $1 \leq q \leq n$, are continuous on $W_{(0,q)}^s(\Omega)$, for $s \geq 0$.*

The proof of Theorem 2 is the same as that of Theorem 1, with the obvious modification in (7)

$$\mathcal{L}_\rho(z_n)(n(z_n), \bar{w}_n) = 2Q_\rho(z_n)(n(z_n), \bar{w}_n) = 0. \quad (10)$$

In the last equality in (10), we have used that w_n is also a null direction of Q_ρ (since $Q_\rho(z_n)(w_n, \bar{w}_n) = \mathcal{L}_\rho(z_n)(w_n, \bar{w}_n)$) and again that Q_ρ is positive semi-definite. Letting n tend to ∞ in (10) gives (2).

Remark 4. In view of the consequences, it would be of interest to have a better understanding of the condition in Theorem 2. An easy observation is that it is satisfied by convex domains (for which global regularity is of course known, [2]). In fact, convex domains satisfy the stronger condition (discussed in the introduction) that the outside sublevel sets of the Euclidean distance are pseudoconvex, see [14], Corollary 2 and the discussion following it.

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